Introduction to Game Theory
Lecture Note 1: Strategic-Form Games and Nash Equilibrium (1)

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Game theory and rationality

- Game theory studies rational players' behavior when they engage in strategic interactions.
- Rationality in preferences:
  - **Completeness**: Between any x and y in a set, either $x \succ y$ (x is preferred to y), or $y \succ x$, or $x \sim y$ (indifferent)
  - **Transitivity**: $x \succeq y$ and $y \succeq z \Rightarrow x \succeq z$ ($\succeq$ means $\succ$ or $\sim$)
- No other restrictions on preferences, e.g., preferences can be selfish or altruistic.
  - But individual rationality does not necessarily mean collective rationality: There can be cycles in group preferences even when all individuals are rational.

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  - But individual rationality does not necessarily mean collective rationality: There can be cycles in group preferences even when all individuals are rational.
- Rationality in choices: The action chosen by a decision maker is better or at least as good as every other available action.

Payoff/utility functions and strategic interaction

- Payoff function/**utility function**: $u(x) \geq u(y)$ iff $x \succeq y$
- For now we only deal with ordinal (as opposed to cardinal) preferences, so you can use many different utility functions to represent the same preference relation.
  - ▷ Say $x \succ y \succ z$. Then $u(x) = 3$, $u(y) = 2$, $u(z) = 1$ represents the same preferences as $u(x) = 100$, $u(y) = 10$, $u(z) = 2$.
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  - ▷ Any strictly increasing transformation of the same utility function will do.
- **Strategic interaction**: A player's payoff depends not only on what she does, but also on what other players do.
- Rational choice in a strategic interaction: The action chosen by a decision maker is better or at least as good as every other available action, *given what everyone else does*.

Types of games

- Games with complete information
  - Static games
  - Dynamic games
- Games with incomplete information
  - Static games (Bayesian games)
  - Dynamic games (dynamic Bayesian games)

Static games of complete information

- **Static games**: simultaneous-move, single-shot games
- **Complete information**: each player knows other players' utility functions
- We use the *strategic/normal form* to represent a static game of complete information
- Definition: A strategic-form game consists of
  1. A set of players
  2. For each player, a set of actions/strategies
  3. For each player, preferences over the set of action/strategy profiles

Static games of complete information

- **Strategy profile**: a list of all players' strategies
    - ▷ E.g, my strategies: easy exam or hard exam; your strategies: study hard or not
    - ▷ Strategy/action profiles: (easy exam, study hard), (easy exam, not study), any other?
- Preferences are over strategy profiles rather than over one's own strategies (whether you want to study hard or not may depend on whether the exam will be easy or hard)
- In single-shot games, actions are equivalent to strategies.

Illustration: a protest/rebellion game

- Players: two citizens, 1 and 2
- Actions for each player: {protest, stay home}
- Outcomes and preferences:
  - ▷ If both protest/rebel, they get a reward (better regime), which outweighs the protest cost; if both stay home, status quo remains; if one protests and the other not, the protest fails and the lone protester pays the cost (and possibly gets punished).
  - ▷ $u_1(\text{protest, protest}) > u_1(\text{home, home}) = u_1(\text{home, protest}) > u_1(\text{protest, home})$
  - ▷ $u_2(\text{protest, protest}) > u_2(\text{home, home}) = u_2(\text{protest, home}) > u_2(\text{home, protest})$

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- Game representation

		Citizen 2	
		Protest	Stay Home
Citizen 1	Protest	$1, 1$	$-1, 0$
	Stay Home	$0, -1$	$0, 0$

Nash equilibrium

- Definition: A strategy profile a^* is a **Nash equilibrium** if, for every player i and every strategy a_i of player i, a^* is at least as good for player i as the strategy profile (a_i, a^*_{-i}) in which player i chooses a_i while every other player j chooses a^*_j.
- In other words: $u_i(a^*) \geq u_i(a_i, a^*_{-i})$ for every strategy a_i of every player i.
- In plain English: *No one can do better by unilaterally deviating from the strategy profile.*
- A Nash equilibrium is a **steady state**. It embodies a stable "social norm:" If everyone else sticks to it, no one has incentive to deviate from it.

The rebellion game

- The Nash equilibrium/equilibria in the rebellion game?

		Citizen 2	
		Rebel	Stay Home
Citizen 1	Rebel	1, 1	−1, 0
	Stay Home	0, −1	0, 0

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- Two strategy profiles are NE: (rebel, rebel) and (stay home, stay home)

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- Two strategy profiles are NE: (rebel, rebel) and (stay home, stay home)
- This is a game with **multiple equilibria**, which characterize a great deal of human interactions
- A coordination game; the starting point of many regime change models

Battle of the sexes

- Now let's look at a few simple, canonical games
- Battle of the sexes (the representation is a bit stereotypical; apologies): He wants to watch soccer, she wants to watch ballet, but they would rather be together than separate

		She	
		Soccer	Ballet
He	Soccer	2, 1	0, 0
	Ballet	0, 0	1, 2

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- BoS can be used to model situations in which players have different (policy) preferences but still want to cooperate

Prisoner's dilemma

- Prisoner's dilemma: perhaps the simplest and best known game in the world, but often misunderstood

		Suspect 2	
		Silent	Confess
Suspect 1	Silent	0, 0	−2, 1
	Confess	1, −2	−1, −1

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- The game has a **unique equibrium**: (confess, confess)
- In PD each player has an **dominant strategy**: a strategy that is better for a player regardless of what other players do

Prisoner's dilemma cont.

- Tragedy of the PD game: There is an outcome that is better for *both* players, but they just cannot achieve it.
- Would communication between the two players help them?
  - Watch a real game: `https://www.youtube.com/watch?v=yM38mRHY150`
- Applications: arms race; tragedy of commons

A variant of the PD game and strict vs. non-strict equilibria

- Recall that if an action profile a^* is a NE, then $u_i(a^*) \geq u_i(a_i, a^*_{-i})$ for every action a_i of every player i.
- An equilibrium is **strict** if each player's equilibrium action is *better* than all her other actions. Or, $u_i(a^*) > u_i(a_i, a^*_{-i})$ for every action $a_i \neq a^*_i$ of player i.

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- A variant of the prisoner's dilemma game

		Player 2	
		Split	Steal
Player 1	Split	5, 5	0, 10
	Steal	10, 0	0, 0

- How many Nash equilibria? Any strict NE?

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  $\Rightarrow$ 3 and 0

Matching pennies

- A purely conflictual game (BoS and PD have elements of cooperation)

		Player 2	
		Head	Tail
Player 1	Head	1, −1	−1, 1
	Tail	−1, 1	1, −1

- Player 1 wants to take the same action as player 2, but player 2 wants to take the opposite action.
- Any (pure-strategy) Nash equilibrium?

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- Player 1 wants to take the same action as player 2, but player 2 wants to take the opposite action.
- Any (pure-strategy) Nash equilibrium?
  - ⇒ No

The chicken game (hawk-dove)

- Two drivers drive towards each other on a single lane. If neither swerves, they collide and may die; if one swerves while the other does not, the one who swerves loses face while the other gains respect.

		Driver 2	
		Straight	Swerve
Driver 1	Straight	$-10, -10$	$1, -1$
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- What are the Nash equilibria?

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- What are the Nash equilibria?
  - $\Rightarrow$ (straight, swerve) and (swerve, straight)
- Application: brinkmanship
- Reducing options in a chicken game: Throwing away the steering wheel? Burning the bridge after crossing the river?

Stag hunt

- Two hunters can succeed in catching a stag if they work together, but each can catch a hare alone.

		Hunter 2	
		Stag	Hare
Hunter 1	Stag	2, 2	0, 1
	Hare	1, 0	1, 1

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- What are the Nash equilibria?
  - ⇒ (stag, stag) and (hare, hare)
- Application: cooperative project in which each has a safe option (e.g., the rebellion game)

Coordination and the focal point

- A pure coordination game: choosing a meeting place

		She	
		Times Square	Statue of Liberty
He	Times Square	1, 1	0, 0
	Statue of Liberty	0, 0	1, 1

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- NE: (TS, TS); (SoL, SoL)

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- NE: (TS, TS); (SoL, SoL)
- **Focal point**: in some real-life situations players may be able to coordinate on a particular equilibrium in a multiple equilibria game, by using information that is abstracted away from the strategic form.
  - ▷ Schelling's experiment about meeting in New York

A thought experiment: driving on Mars

- Imagine there are roads on Mars and two first-time human visitors are driving there. They start on the opposite ends of a two-lane road, not knowing the other driver's background (e.g., national origin). There are no other people including police on Mars. Should they drive on the left or right to avoid collision?

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- NE: (L, L); (R, R)
- The **strategic uncertainty** present in multiple equilibria is perhaps a crucial feature distinguishing the human world from the natural world.

Driving on Mars and the role of institutions

- Now imagine there is a sign (perhaps left by previous visitors) on both ends of the road that says “Drive on the right”, and both drivers know each other can see the sign. How should they drive?

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- The sign creates a focal point and coordinates the drivers’ expectation of how each other will drive.

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- They will likely follow the sign. But why should they follow the sign given that it will not be enforced by anyone?
- The sign creates a focal point and coordinates the drivers' expectation of how each other will drive.
- **Institutions** (laws) are just some ink on paper, but they *can be effective by serving as focal points and change people's expectations about each other's behavior* (Basu 2020; Myerson 2004).

A game of public good provision

Osborne (2004) exercise 33.1: Each of n people chooses whether to contribute a fixed amount toward the provision of a public good. The good is provided iff at least k people contribute, where $2 \leq k \leq n$; if it is not provided, contribution are not refunded. Each person ranks outcomes from best to worst as follows: (a) any outcome in which the good is provided and she does not contribute; (b) any outcome in which the good is provided and she contributes; (c) any outcome in which the good is not provided and she does not contribute; (d) any outcome in which the good is not provided and she contributes. Formulate this situation as a strategic game and find the NE.

A game of public good provision: strategic form

- Players: the n people
- Each player's set of actions: {contribute, not contribute}
- Preferences: $u_i(a) > u_i(b) > u_i(c) > u_i(d)$

A game of public good provision: NE

- Is there a NE in which more than k people contribute? One in which k people contribute? One in which fewer than k contribute?

A game of public good provision: NE

- Is there a NE in which more than k people contribute? One in which k people contribute? One in which fewer than k contribute?
- NE: k people contribute; none contributes