Introduction to Game Theory
Lecture Note 2: Strategic-Form Games and Nash Equilibrium (2)

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Best response functions: example

- In simple games we can examine each action profile in turn to see if it is a Nash equilibrium. In more complicated games it is better to use "best response functions."
- Example:

		Player 2		
		L	M	R
Player 1	T	1, 1	1, 0	0, 1
	B	1, 0	0, 1	1, 0

- What are player 1's best response(s) when player 2 chooses L, M, or R?

Best response functions: definition

- Notation:
  $B_i(a_{-i}) = \{a_i \text{ in } A_i : U_i(a_i, a_{-i}) \geq U_i(a_i', a_{-i}) \text{ for all } a_i' \text{ in } A_i\}$.
- I.e., any action in $B_i(a_{-i})$ is *at least as good* for player i as every other action of player i when the other players' actions are given by a_{-i}.
- Example:

		Player 2		
		L	M	R
Player 1	T	1, 1	1, 0	0, 1
	B	1, 0	0, 1	1, 0

- $B_1(L) = \{T, B\}$, $B_1(M) = \{T\}$, $B_1(R) = \{B\}$

Using best response functions to define Nash equilibrium

- Definition: the action/strategy profile a^* is a Nash equilibrium of a strategic game if and only if every player's action is a best response to the other players' actions: a_i^* is in $B_i(a_{-i}^*)$ for every player i.
- If each player has a single best response to each list a_{-i} of the other players' actions, then $a_i = b_i(a_{-i}^*)$ for every i.

Using best response functions to find Nash equilibrium

- Method:
  - find the best response function of each player
  - find the action profile in which each player's action is a best response to the other player's action
- Example:

		Player 2		
		L	M	R
Player 1	T	1, 2	2, 1	1, 0
	M	2, 1	0, 1	0, 0
	B	0, 1	0, 0	1, 2

One more example

- Osborne (2004) exercise 39.1: Two people are involved in a synergistic relationship. If both devote more effort to the relationship, they are both better off. For any given effort of individual j, the return to individual i's effort first increases, then decreases. Specifically, an effort level is a non-negative number, and each individual i's preferences are represented by the payoff function $u_i = e_i(c + e_j - e_i)$, where e_i is i's effort level, e_j is the other individual's effort level, and $c > 0$ is a constant.

Solving the example

- $u_i = -e_i^2 + (c + e_j)e_i$, a quadratic function; inverted U-shape
- $u_i = 0$ if $e_i = 0$ or if $e_i = c + e_j$, so anything in between will give i a positive payoff
- Symmetry of quadratic functions means that $b_i(e_j) = \frac{1}{2}(c + e_j)$
- Similarly, $b_j(e_i) = \frac{1}{2}(c + e_i)$
- In equilibrium, therefore, $e_i = \frac{1}{2}(c + e_j)$ and $e_j = \frac{1}{2}(c + e_i)$; solving the two equations together yield that $e_i^* = e_j^* = c$.

Alternatively, we can use (just) a little calculus

- Maximize $u_i = e_i(c + e_j - e_i)$
- First order condition: $\frac{\partial u_i}{\partial e_i} = c + e_j - 2e_i = 0 \Rightarrow$

$$e_i = \frac{c + e_j}{2} \tag{1}$$

- Similarly,

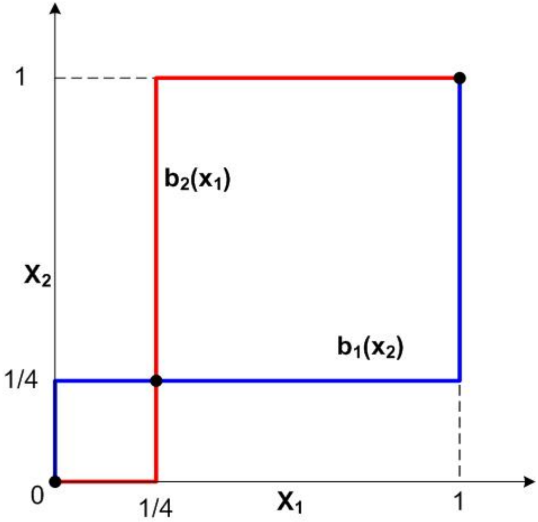
$$e_j = \frac{c + e_i}{2} \tag{2}$$

- Plugging (2) into (1), we know $e_i^* = e_j^* = c$.

Another example about cooperation

- Osborne (2004) exercise 42.2(b): Two people are engaged in a joint project. If each person i puts in effort x_i, a non-negative number equal to at most 1, which costs her x_i, each person will get a utility $4x_1x_2$. Find the NE of the game. Is there a pair of effort levels that yields higher payoffs for both players than do the NE effort levels?

Best response functions in graph



Oligopolistic competition: the Cournot model

- Two firms produce the same product. The unit cost of production is c. Let q_i be firm i's output, $Q = \sum_{i=1}^{2} q_i$, then the market price P is $P(Q) = \alpha - Q$, where α is a constant.
- Firms choose their output simultaneously. What is the NE?
- Each firm wants to maximize profit. Firm 1's profit is

$$\begin{aligned}\pi_1 &= P(Q)q_1 - cq_1 \\ &= (\alpha - q_1 - q_2)q_1 - cq_1.\end{aligned}$$

- Differentiate π_1 with respect to q_1, we know by the first order condition that firm 1's optimal output (best response) is

$$q_1 = b_1(q_2) = \frac{\alpha - q_2 - c}{2} \tag{3}$$

The Cournot model (cont.)

- Similarly (since the game is symmetric), firm 2's optimal output is

$$q_2 = b_2(q_1) = \frac{\alpha - q_1 - c}{2} \tag{4}$$

- Solving equations (3) and (4) together, we have

$$q_1^* = q_2^* = \frac{1}{3}(\alpha - c).$$

- If the two firms can collude, they would maximize $PQ - cQ = (\alpha - Q)Q - cQ$. The output would be $Q = \frac{1}{2}(\alpha - c) < \frac{2}{3}(\alpha - c)$, and the market price would be $\alpha - Q = \alpha - \frac{1}{2}(\alpha - c) > \alpha - \frac{2}{3}(\alpha - c)$.
- Competition (instead of collusion) increases total output, and reduces market price.

The strategic model of the war of attrition

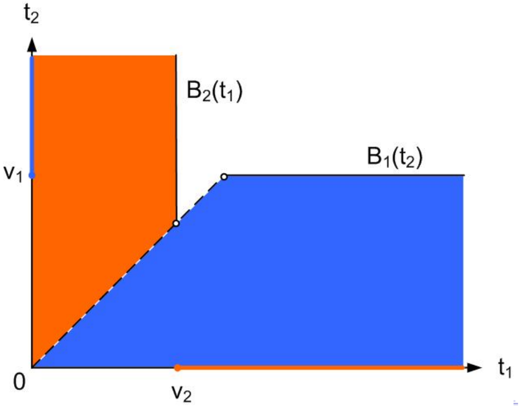
- Examples: animals fighting over prey; interest groups lobbying against each other; countries fighting each other to see who will give up first...
- Model setup
    - ▷ Two players, i and j, vying for an object, which is respectively worth v_i and v_j to the two players; a 50% chance of obtaining the object is respectivley worth $\frac{v_i}{2}$ and $\frac{v_j}{2}$.
    - ▷ Time starts at 0 and runs indefinitely; each unit of time that passes before one of the parties concedes costs each player one unit of utility.
    - ▷ So, a player i's utility is

$$u_i(t_i, t_j) = \begin{cases} -t_i, & \text{if } t_i < t_j; \\ \frac{1}{2}v_i - t_j, & \text{if } t_i = t_j; \\ v_i - t_j, & \text{if } t_i > t_j. \end{cases}$$

Best response function

- Player 2's best response function is (orange)

$$B_2(t_1) = \begin{cases} \{t_2 : t_2 > t_1\}, & \text{if } t_1 < v_2; \\ \{t_2 : t_2 = 0 \text{ or } t_2 > t_1\}, & \text{if } t_1 = v_2; \\ \{0\}, & \text{if } t_1 > v_2. \end{cases}$$



NE in war of attrition

- (t_1, t_2) is a NE iff $t_1 = 0$ and $t_2 \geq v_1$, or $t_2 = 0$ and $t_1 \geq v_2$.
- In equilibrium, either player may concede first, including the one who values the object more.
- The equilibria are asymmetric, even when $v_1 = v_2$ (i.e., when the game is symmetric).
- A game is symmetric if $u_1(a_1, a_2) = u_2(a_2, a_1)$ for every action pair (a_1, a_2) (if you and your opponent exchange actions, you also exchange your payoffs).

A direct argument

- If $t_i = t_j$, then either player can increase her payoff by conceding slightly later and obtaining the object for sure; $v_i - t_i - \epsilon > \frac{1}{2}v_i - t_i$ for a sufficiently small ϵ.
- If $0 < t_i < t_j$, player i should rather choose $t_i = 0$ to reduce the loss.
- If $0 = t_i < t_j < v_i$, player i can increase her payoff by conceding slightly after t_j, but before $t_i = v_i$.
- The remaining case is $t_i = 0$ and $t_j \geq v_i$, which we can easily verify as a NE.

Domination

- Player *i*'s action a_i' **strictly dominates** action a_i'' if

$$u_i(a_i', a_{-i}) > u_i(a_i'', a_{-i})$$

  for every list a_{-i} of the other players' actions. In this case the action a_i'' is **strictly dominated**.
- In Prisoner's Dilemma, "confess" strictly dominates "silent".

		Suspect 2	
		Silent	Confess
Suspect 1	Silent	0, 0	−2, 1
	Confess	1, −2	−1, −1

- If player *i*'s action a_i' strictly dominates every other action of hers, then a_i' is *i*'s **strictly dominant action**.

Elimination of strictly dominated action

- Not every game has a strictly dominated action. But if there is, it is not used in any Nash equilibrium and so can be eliminated.
- Any strictly dominated action in the following game? Any strictly dominant action?

		Player 2		
		L	C	R
Player 1	U	7, 3	0, 4	4, 4
	M	4, 6	1, 5	5, 3
	D	3, 8	0, 2	3, 0

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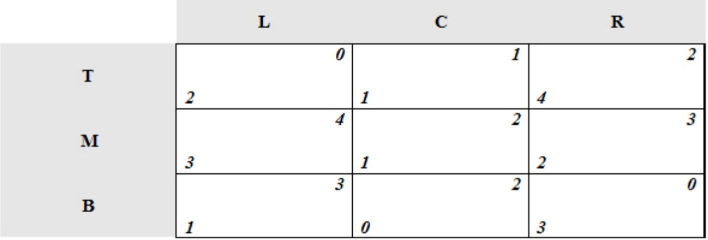
$\Rightarrow$ D is strictly dominated by M

Iterated elimination of strictly dominated action

- Sometimes we can repeat the procedure: eliminate all strictly dominated actions, and then continue to eliminate strategies that are now dominated in the simpler game.
- Are there more than one actions that can be eliminated from the following game?



	L	C	R
T	0 2	1 1	2 4
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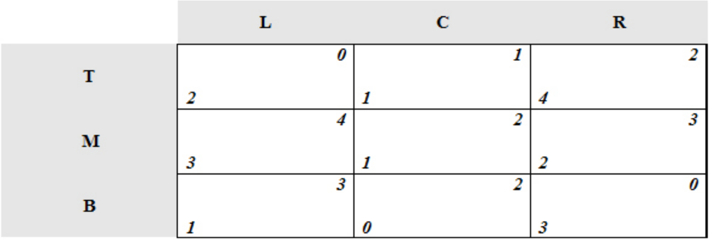

	L	C	R
T	0 2	1 1	2 4
M	4 3	2 1	3 2
B	3 1	2 0	0 3

⇒ First B and then C can be eliminated

Weak Domination

- Player i's action $a_i^{'}$ **weakly dominates** action $a_i^{''}$ if

$$u_i(a_i^{'}, a_{-i}) \geq u_i(a_i^{''}, a_{-i})$$

  for every list a_{-i} of the other players' actions, and

$$u_i(a_i^{'}, a_{-i}) > u_i(a_i^{''}, a_{-i})$$

  for some list a_{-i} of the other players' actions.
- Action $a_i^{''}$ is then **weakly dominated**.

Weak Domination

- Any weakly dominated action in the following game?

Player 2

Player 1		L	C	R
	U	7, 3	0, 4	4, 4
Player 1	M	4, 6	1, 5	5, 3
	D	3, 8	1, 2	4, 0

Weak Domination

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⇒ R weakly dominated by C; D weakly dominated by M

Weak Domination

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$\Rightarrow$ R weakly dominated by C; D weakly dominated by M

- If player *i*'s action a_i' weakly dominates every other action of hers, then a_i' is *i*'s **weakly dominant action**.

Example: Voting

There are two candidates A and B for an office, and N voters, $N \geq 3$ and odd. A majority of voters prefer A to win.

- Is there a strictly dominated action? A weakly dominated action?
- What are the Nash equilibria of the game? Hint: Let N_A denote the number of voters that vote for A, and N_B the number of voters that vote for B, $N_A + N_B = N$, then
  - What if $N_A = N_B + 1$ or $N_B = N_A + 1$, and some citizens who vote for the winner actually prefer the loser?
  - What if $N_A = N_B + 1$ or $N_B = N_A + 1$, and nobody who votes for the winner actually prefers the loser?
  - Can it happen that $N_A = N_B + 2$ or $N_B = N_A + 2$?
  - What if $N_A \geq N_B + 3$ or $N_B \geq N_A + 3$?

Solving the voting problem

- What if $N_A = N_B + 1$ or $N_B = N_A + 1$, and some citizens who vote for the winner actually prefer the loser? $\Rightarrow$ Such a citizen can unilaterally deviate and make her favorite candidate win. Not a NE.
- What if $N_A = N_B + 1$ or $N_B = N_A + 1$, and nobody who votes for the winner actually prefers the loser? $\Rightarrow$ The former is a NE, but the latter cannot occur (the supporters of B would be more than half).
- Can it be happen that $N_A = N_B + 2$ or $N_B = N_A + 2$? $\Rightarrow$ No, because N is odd.
- What if $N_A \geq N_B + 3$ or $N_B \geq N_A + 3$? $\Rightarrow$ Yes, NE.

Strategic voting

- There are three candidates, A, B, and C, and no voter is indifferent between any two of them.
- Voting for one's least preferred candidate is a weakly dominated action. What about voting for one's second preference? Not dominated.

Strategic voting

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- Voting for one's least preferred candidate is a weakly dominated action. What about voting for one's second preference? Not dominated.
- Suppose you prefer A to B to C, and the other citizens' votes are tied between B and C, with A being a distant third. Then voting for B, your second preference, is your best choice! ⇒ **strategic voting**
- In two-candidate elections you are weakly better off by voting for your favorite candidate, but in three-candidate elections that is not necessarily the case. E.g, Nader supporters in the 2000 US election.

Hotelling/Downsian model

- A workhorse model of electoral competition. First proposed by Hotelling (1929) and popularized by Downs (1957).
- Setup:
  - ▷ Parties/candidates compete by choosing a policy on the line segment $[0, 1]$. The party with most votes wins; if there is a tie, the parties that tie have the same probability of winning.
  - ▷ Parties only care about winning, and will commit to the platforms they have chosen.
  - ▷ Each voter has a favorite policy on $[0, 1]$; her utility decreases as the winner's position is further away from her favorite policy $\Rightarrow$ **single-peaked** preference
  - ▷ Each voter will vote **sincerely**, choosing the party whose position is closest to her favorite policy.
  - ▷ There is a median voter position, m.

Two parties

- Suppose there are 2 parties, L and R. What is the Nash equilibrium for the parties' positions?

Two parties

- Suppose there are 2 parties, L and R. What is the Nash equilibrium for the parties' positions?
- The unique equilibrium is both parties choose position m.
  - ▷ (m, m) is clearly a NE
  - ▷ any other action profile is not a NE
- This is the **Median Voter Theorem**.

Three parties

- Suppose there is a continuum of voters, with favorite policies uniformly distributed on $[0, 1]$, and the number of parties is 3 (L, C, R). Do we still have the equilibrium that all parties choose m?

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- Would the three parties positioning at 0.45, 0.55, 0.6 be a NE?

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- Would the three parties positioning at 0.45, 0.55, 0.6 be a NE?
  - ⇒ Yes. L wins already; C and R cannot win by moving anywhere.

Condorcet winner

- A **Condorcet winner** in an election is a position, x^*, such that for every other position y that is different from x^*, a majority of voters prefer x^* to y.
- The median voter position is a Condorcet winner.
- Not all election games have a Condorcet winner.
  - ▷ Condorcet paradox: A prefers X to Y to Z; B prefers Y to Z to X; C prefers Z to X to Y.
- Even if there is a Condorcet winner, it only has guaranteed victory in pairwise comparisons, not necessarily when there are three or more policy alternatives.
  - ▷ E.g., uniform distribution of voter preferences, sincere voting, candidate A = .3, B = .6, C = .7