Introduction to Game Theory
Lecture Note 3: Mixed Strategies

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Mixed strategies and von Neumann-Morgenstern preferences

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- Now we consider **mixed strategy** equilibria, in which players can randomize over their actions.

Mixed strategies and von Neumann-Morgenstern preferences

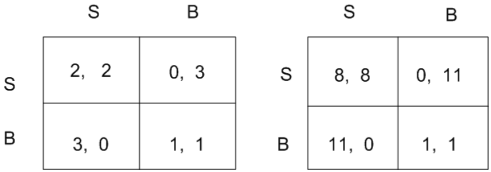
- So far what we have considered are **pure strategy** equilibria, in which players choose deterministic actions.
- Now we consider **mixed strategy** equilibria, in which players can randomize over their actions.
- This means we need to deal with preferences regarding lotteries, i.e., the **vNM preferences**–*preferences regarding lotteries over action profiles that may be represented by the expected value of a payoff function over action profiles.*
- Say your preference ordering is $A \succ B \succ C$. Given two lotteries: a) A occuring with probability .9 and C occuring with probability .1, and b) A occuring with probability .5 and B occuring with probability .5. Which do you prefer?

Mixed strategies and von Neumann-Morgenstern preferences

- In pure strategy equilibria, we deal with **ordinal preferences**, which only specify the order of your preferences, not how much you prefer one item over another.
- With **vNM preferences**, the payoff numbers in a game state the intensity of your preferences, not just the order, and you can take expectations over the numbers.

von Neumann-Morgenstern Expected Utilities

- The following tables represent the same game with ordinal preferences but different games with vNM preferences.



	S	B
S	2, 2	0, 3
B	3, 0	1, 1

	S	B
S	8, 8	0, 11
B	11, 0	1, 1

- With vNM preferences, we can derive **expected utilities** over lotteries: $U(p_1, ..., p_K) = \sum_{k=1}^{K} p_k u(a_k)$, where a_k is the kth outcome of the lottery, and p_k is the probability that a_k will happen.
- You prefer the lottery $(p_1,..., p_K)$ to the lottery $(p'_1, ..., p'_K)$ only if $\sum_{k=1}^{K} p_k u(a_k) > \sum_{k=1}^{K} p'_k u(a_k)$.

Mixed strategy Nash equilibrium

- A **mixed strategy** of a player in a strategic game is a probability distribution over the player's actions, denoted by $\alpha_i(a_i)$; e.g., $\alpha_i(\text{left}) = \frac{1}{3}, \alpha_i(\text{right}) = \frac{2}{3}$.
  - A pure strategy is a mixed strategy that assigns probability 1 to a particular action.

Mixed strategy Nash equilibrium

- A **mixed strategy** of a player in a strategic game is a probability distribution over the player's actions, denoted by $\alpha_i(a_i)$; e.g., $\alpha_i(\text{left}) = \frac{1}{3}, \alpha_i(\text{right}) = \frac{2}{3}$.
  - ▷ A pure strategy is a mixed strategy that assigns probability 1 to a particular action.
- The mixed strategy profile α^* in a strategic game is a **mixed strategy Nash equilibrium** if

$$U_i(\alpha_i^*, \alpha_{-i}^*) \geq U_i(\alpha_i, \alpha_{-i}^*), \forall \alpha_i \text{ and } i,$$

  where $U_i(\alpha)$ is player i's expected payoff with the mixed strategy profile α.
- Using best response functions, α^* is a mixed strategy NE iff α_i^* is in $B_i(\alpha_i^*)$ for every player i.

Matching Pennies reconsidered

- There is no pure strategy Nash equilibrium in Matching Pennies.

		Player 2	
		Head	Tail
Player 1	Head	$1, -1$	$-1, 1$
	Tail	$-1, 1$	$1, -1$

- But there is a mixed strategy NE for the game with the above vNM preferences: $((\text{head}, \frac{1}{2}; \text{tail}, \frac{1}{2}), (\text{head}, \frac{1}{2}; \text{tail}, \frac{1}{2}))$.
- **Theorem** (Nash 1950): Every finite strategic game with vNM preferences has a mixed strategy Nash equilibrium.

Strict domination with mixed strategies

- Player i's mixed strategy α_i strictly dominates her action a'_i if $U_i(\alpha_i, a_{-i}) > u_i(a'_i, a_{-i})$ for every list a_{-i} of the other players' actions. a'_i is strictly dominated.
- In the following game, player 1 has no action that is dominated by a pure strategy. But action T is dominated by the mixed strategy $(M, p; B, 1-p)$, with $\frac{1}{4} < p < \frac{2}{3}$.

Player 2

		L	R
	T	1, a	1, b
Player 1	M	4, c	0, d
	B	0, e	3, f

- A strictly dominated action is not used in any mixed strategy Nash equilibrium.

Weak domination with mixed strategies

- Player i's mixed strategy α_i weakly dominates action a'_i if $U_i(\alpha_i, a_{-i}) \geq u_i(a'_i, a_{-i})$ for every list a_{-i} of the other players' actions, and $U_i(\alpha_i, a_{-i}) > u_i(a'_i, a_{-i})$ for some list a_{-i} of the other players' actions.
- A weakly dominated action, however, may be used in a mixed strategy NE.
- But *every finite strategic game has a mixed strategy NE in which no player's strategy is weakly dominated.*

Characterization of mixed strategy NE in finite games

A characterization for finite strategic games: a mixed strategy profile α^* is a mixed strategy NE iff, for each player *i*,

1. the expected payoff, given α^*_{-i}, to every action to which α^*_i assigns positive probability is the same ($\Rightarrow$ otherwise *i* should just play the more profitable action rather than mixing it with other actions);
   - ▷ In other words, other players' equilibrium mixed strategies keep you indifferent between a set of your actions.
2. the expected payoff, given α^*_{-i}, to every action to which α^*_i assigns zero probability is lower or at most equal to the expected payoff to any action to which α^* assigns positive probability ($\Rightarrow$ otherwise *i* should play that action).

Method for finding all mixed strategy NE

1. Eliminate strictly dominated actions from the game
2. For each player i, choose a subset S_i of her set A_i of actions
3. Check if there is a mixed strategy profile α that (1) assigns positive probability only to actions in S_i, and (2) satisfies the two conditions in the previous characterization
4. Repeat the analysis for every other collection of subsets of the players' sets of actions

Example

- Consider the variant of the Battle of Sexes game below. What are the mixed strategy NE?

		Player 2		
		B	S	X
Player 1	B	4, 2	0, 0	0, 1
	S	0, 0	2, 4	1, 3

- By inspection, we see there is no dominated strategy to be eliminated. Further, (B, B) and (S, S) are two pure strategy equilibria.

Four possible kinds of mixed strategy equilibrium

- What about an equilibrium in which player 1 plays a pure strategy (B or S), while player 2 plays a strictly mixed strategy? Condition 1 of the characterization impossible to meet.
- Similar reasoning rules out the potential equilibrium in which player 2 plays pure strategy while player 1 randomize over her two actions.
- What about an equilibrium in which player 1 mixes over her two actions, while player 2 mixes over two of her three actions: B &S, B &X, or S&X?
- What about an equilibrium in which player 1 mixes over her two actions, and player 2 mixes over her three actions?

Analyzing the example (1)

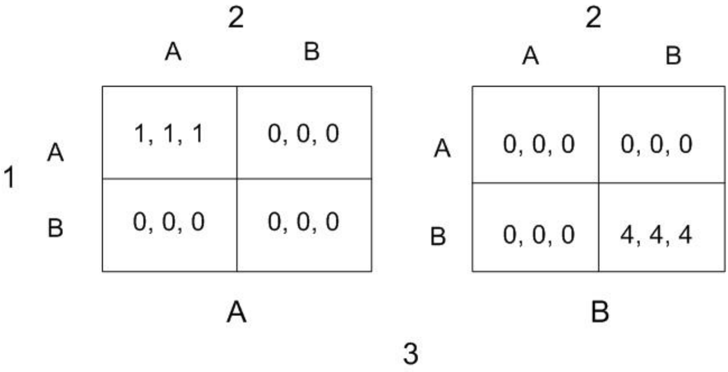
- Let player 1's probability of playing B be p (hence $1-p$ for S).
- Player 2 mixes over B and S:
  - ▷ To satisfy conditions 1 and 2 we need $2p = 4(1-p) \geq p + 3(1-p)$. Impossible to hold.
- Player 2 mixes over B and X:
  - ▷ To satisfy the two conditions we need $2p = p + 3(1-p) \geq 4(1-p) \Rightarrow p = \frac{3}{4}$.
  - ▷ Next we should examine player 2's randomization. Let q be her probability of choosing B (hence $1-q$ for X). For player 1 to be indifferent between her two actions (condition 1; condition 2 moot here), $4q + 0 = 0 + (1-q) \Rightarrow q = \frac{1}{5}$.
  - ▷ Thus $((B, \frac{3}{4}; S, \frac{1}{4}), (B, \frac{1}{5}; S, 0; X, \frac{4}{5}))$ is a mixed strategy NE.

Analyzing the example (2)

- Player 2 mixes over S and X:
  - ▷ In this case player 1 will always choose S. So no NE in which player 1 mixes over B and S.
- Player 2 mixes over B, S, and X
  - ▷ For player 2 to be indifferent between her three actions (condition 1; condition 2 moot here)), we need $2p = 4(1-p) = p + 3(1-p) \Rightarrow$ impossible.
- The NE are the two pure strategy equilibria and the strictly mixed strategy NE $\left((B, \frac{3}{4}; S, \frac{1}{4}), (B, \frac{1}{5}; S, 0; X, \frac{4}{5})\right)$.

A three-player example

- Player 1 chooses between rows, player 2 chooses between columns, and player 3 chooses between tables.



		2	
		A	B
1	A	1, 1, 1	0, 0, 0
	B	0, 0, 0	0, 0, 0

A

	2	
	A	B
A	0, 0, 0	0, 0, 0
B	0, 0, 0	4, 4, 4

B

3

Analyzing the three-player example

- By inspection (A, A, A) and (B, B, B) are two pure-strategy NE.
- If one of the players' strategy is pure, obviously the other two should choose the first player's action rather than mix over two or more actions.
- The only remaining case is all three mix over A and B. Let p, q, and r respectively denote the three players' probability of choosing A. Then condition 1 of the characterization requires
  1. $qr = 4(1-q)(1-r)$;
  2. $pr = 4(1-p)(1-r)$;
  3. $pq = 4(1-p)(1-q)$.
- Therefore $p = q = r = \frac{2}{3}$ is a mixed strategy NE.

Characterization of mixed strategy NE in infinite games

- Finite games must have a mixed strategy NE. Infinite games may or may not have one.
- Condition 1 of the characterization in finite games does not apply in infinite games because the probabilities are now assigned to sets of actions, not single actions.
- **A characterization for infinite strategic games**: a mixed strategy profile α^* is a mixed strategy NE iff, for each player i,
  - ▷ for no action a_i does the action profile (a_i, α^*_{-i}) yield player i an expected payoff greater than her expected payoff to α^*;
  - ▷ α^* assigns probability zero to the set of actions a_i for which the action profile (a_i, α^*_{-i}) yields player i an expected payoff less than her expected payoff from α^*.