Introduction to Game Theory
Lecture Note 6: Bargaining

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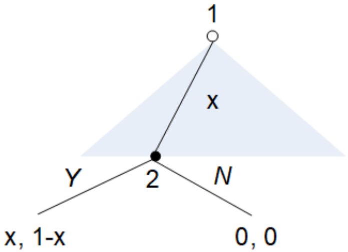
The ultimatum game again

- Recall the ultimatum game: two people need to decide how to divide a dollar. Player 1 proposes to give herself x and give $(1-x)$ to player 2. If player 2 accepts the offer, then they respectively receive x and $1-x$. If 2 rejects the offer, then neither person receives anything.



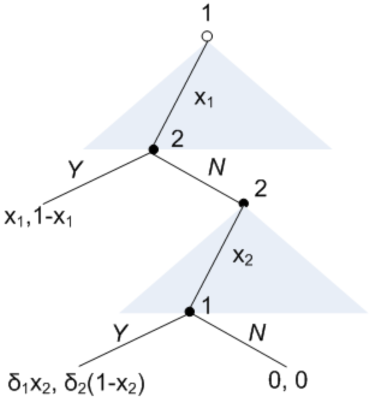
- The stark result of the game is due to the fact that player 2 has no bargaining power.

A finite horizon game with alternating proposals

- Now suppose a player can make a counter proposal after rejecting the other player's proposal; but they have to reach an agreement before or at period $T < \infty$. If the proposal made in period T is rejected, the game ends with both players getting 0.
- Further, each player i discounts the future by a discount factor δ_i; a deal reached in period t that gives player i a share of s_i is equivalent to giving her $\delta_i^{t-1} s_i$ today.
- What will be the equilibrium outcome?

$T = 2$

- First, consider the case with $T = 2$, as in the following, where in period t the offer is $(x_t, 1 - x_t)$, i.e., player 1 gets x_t.



- Note the discount factors in the above graph. What is the SPNE? Use backward induction.

Solution for $T = 2$

- The subgame starting after a history in which player 2 rejected the initial offer by player 1 is just the standard ultimatum game, and so we know in a SPNE player 2 will offer zero to the other player and give herself one, which will be accepted by player 1.
- Given this, at the beginning of the game player 1 has to make an offer $\geq \delta_2$ to player 2 in order for player 2 to accept the offer. Why?
- The unique SPNE is: (1) $x_1 = 1 - \delta_2$, (2) player 2 accepts any initial offer that gives her $\geq \delta_2$ and rejects any other offer, (3) if player 2 rejects player 1's offer, player 2 offers zero to player 1 in period 2, and (4) player 1 accepts all offers proposed by player 2 in period 2.

An arbitrary finite T

- Consider a general T, $T < \infty$. Assume player 2 makes the proposal in period T, i.e., T is even. (What if T odd? Similar logic.)
- In period T player 2 will make the proposal $(0, 1)$.
- Therefore in the penultimate period $(T - 1)$, player 1's proposal is $(1 - \delta_2, \delta_2)$. Therefore in period $T - 2$, player 2's proposal is $(\delta_1(1 - \delta_2), 1 - \delta_1(1 - \delta_2))$ ...
- In the first period, then, player 1 proposes that she gets a share equal to $x_1 = 1 - \delta_2(1 - \delta_1(1 - \delta_2(...)))$ and leave the rest to player 2.

$T < \infty$ cont.

- In equilibrium, player 1 will get

$$
\begin{aligned}
x_1^* &= 1 - \delta_2(1 - \delta_1(1 - \delta_2(\ldots))) \\
&= 1 - \delta_2 + \delta_1\delta_2 - \delta_1\delta_2^2 + \ldots + \delta_1^{T/2-1}\delta_2^{T/2-1} - \delta_1^{T/2-1}\delta_2^{T/2} \\
&= \sum_{t=0}^{T/2-1} (1-\delta_2)(\delta_1\delta_2)^t \\
&= \frac{(1-\delta_2)[1-(\delta_1\delta_2)^{T/2}]}{1-\delta_1\delta_2}.
\end{aligned}
$$

- Player 2 will get

$$1 - x_1^* = \frac{\delta_2(1-\delta_1) + (1-\delta_2)(\delta_1\delta_2)^{T/2}}{1-\delta_1\delta_2}.$$

Infinite Horizon Bargaining: The Rubinstein Model

- If T is infinity, the game is much harder since we cannot use backward induction.
- Rubinstein (1982), however, shows that the solution has a remarkably simple form.
- **Proposition**: The alternating-proposal bargaining game with $T = \infty$ has a unique SPNE: in any period in which player i makes a proposal, she proposes her own share to be

$$\frac{1-\delta_j}{1-\delta_i\delta_j},$$

  and the other player j's share to be

$$\frac{\delta_j(1-\delta_i)}{1-\delta_i\delta_j}.$$

  Player j accepts this or any higher offer and rejects any lower offer.

The Rubinstein Model: Proof (1)

- Let $\underline{v}_i$ be the lowest payoff player i receives in any SPNE *in a subgame when she makes the initial offer* and let $\overline{v}_i$ be her highest SPNE payoff in such a subgame. (Likewise, we have $\underline{v}_j$ and $\overline{v}_j$).
- Consider a subgame in which i makes the initial offer. Player j will not accept an offer that gives her less than $\delta_j \underline{v}_j$, so

$$\overline{v}_i \leq 1 - \delta_j \underline{v}_j. \tag{1}$$

- Similarly, j will accept any offer that gives her at least $\delta_j \overline{v}_j$, so

$$\underline{v}_i \geq 1 - \delta_j \overline{v}_j. \tag{2}$$

- By symmetry, j's payoffs should satisfy

$$\overline{v}_j \leq 1 - \delta_i \underline{v}_i \tag{3}$$

and

$$\underline{v}_j \geq 1 - \delta_i \overline{v}_i. \tag{4}$$

The Rubinstein Model: Proof (2)

- Subtracting (2) from (1) and (4) from (3), we have

$$\overline{v}_i - \underline{v}_i \leq \delta_j(\overline{v}_j - \underline{v}_j) \tag{5}$$

and

$$\overline{v}_j - \underline{v}_j \leq \delta_i(\overline{v}_i - \underline{v}_i). \tag{6}$$

- Multipling (6) through by δ_j and combining this with (5) gives

$$\overline{v}_i - \underline{v}_i \leq \delta_i\delta_j(\overline{v}_i - \underline{v}_i).$$

- Since $\delta_i\delta_j < 1$, this implies $\overline{v}_i = \underline{v}_i \equiv v_i$.
- Doing the same thing w.r.t. j's payoffs yields $\overline{v}_j = \underline{v}_j \equiv v_j$. Hence, SPNE payoffs are unique.

The Rubinstein Model: Proof (3)

- Since SPNE payoffs are unique, (1) and (2) become

$$v_i = 1 - \delta_j v_j,$$

and (3) and (4) become

$$v_j = 1 - \delta_i v_i.$$

- Direct substitution then yields

$$v_i^* = \frac{1 - \delta_j}{1 - \delta_i \delta_j} \text{ and } 1 - v_i^* = \frac{\delta_j (1 - \delta_i)}{1 - \delta_i \delta_j}.$$

- Since $1 - v_i^*$ equals $\delta_j v_j^*$, j is indifferent and accepts. Q.E.D.

Properties of the SPNE of Rubinstein Model

- Efficiency: player 2 would accept player 1's first proposal, resulting in immediate agreement without delay (which is costly due to discounting).
- The more patient a player is, the better off she will be: a higher δ_j reduces v_i^* and increases $1 - v_i^*$.
- First mover advantage: if $\delta_i = \delta_j = \delta$, then in SPNE whoever makes the initial proposal offers herself a higher share: $\frac{1-\delta}{1-\delta^2} > \frac{\delta(1-\delta)}{1-\delta^2}$.
    - When the real time between proposals is shortened toward zero, δ approaches 1 ($\delta^{1/n} \to 1$ as $n \to \infty$), and the first mover advantage disappears. Each player will get $1/2$.

Extension A: fixed delay costs per period (1)

- Now suppose there is no discounting, but each player i loses c_i during each period of delay. Player 1 moves first.
- What is the SPNE outcome if $c_1 < c_2$?
- Will player 1 accept any offer $(1-x, x)$ with $x > c_2$? No, player 1 will reject any such offer by player 2. Why?
- In the next round player 1 can propose to give $x - c_2$ to player 2, and player 2 will accept.
- Player 1 is better off with delay because $1 - (x - c_2) - c_1 > 1 - x$.
- Therefore there is no SPNE in which $x > c_2$. In a SPNE $x \leq c_2$.

Fixed delay costs per period (2)

- If the equilibrium offer player 2 will get is $\leq c_2$, then player 2 will accept any offer. Because by rejecting an offer, what player 2 will get is less than (or at most equal to) her delay cost.
- Therefore the SPNE is:
  - player 1 always proposes $(1, 0)$ and accepts a proposal $(y, 1-y)$ iff $y \geq 1 - c_1$;
  - player 2 always proposes $(1 - c_1, c_1)$ and accept all proposals.
- If $c_1 > c_2$, then the subgame following player 2's move (after 2 rejects 1's initial offer) will be exactly like the above, with player 2 making the offer $(0, 1)$.
- Therefore if $c_1 > c_2$, player 1 will propose $(c_2, 1 - c_2)$, and 2 will accept it.

Extension B: risk of breakdown

- Suppose that after any proposal is rejected, there is an exogenous probability $p > 0$ that the negotiation terminates, in which case the two players' payoff are b_1 and b_2.
- To simplify things, assume there is no discounting.
- Let player 1's SPNE proposal when it's her move to propose be $(v_1, 1 - v_1)$, and let player 2's SPNE proposal when it's her move to propose be $(1 - v_2, v_2)$. Solve for v_1 and v_2.
- For player 2 to accept player 1's proposal, we must have

$$1 - v_1 = pb_2 + (1 - p)v_2.$$

- Similarly for player 1 to accept player 2's proposal, we must have

$$1 - v_2 = pb_1 + (1 - p)v_1.$$

- Solving the two equations together yields

$$v_1 = \frac{1 - b_2 + (1 - p)b_1}{2 - p} \text{ and } v_2 = \frac{1 - b_1 + (1 - p)b_2}{2 - p}.$$

Extension C: outside options

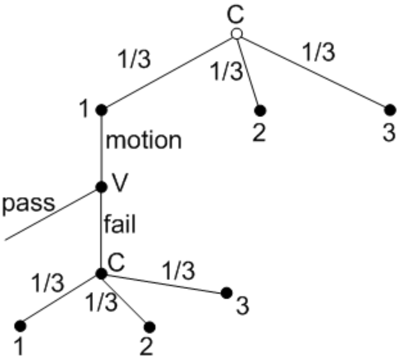
- Suppose that one of the players, say player 2, has the option of leaving the negotiation for an outside option with a fixed payoff x_2.
- Assume that when this happens, player 1's payoff is 0; otherwise the game is similar to the standard Rubinstein model, in which player 2 will accept an offer iff it is $\geq v_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$.
- If $x_2 < v_2^*$, then the outside option is irrelevant, since it is better for player 2 not to opt out.
- If $x_2 > v_2^*$, then the game has a unique SPNE in which
  - player 1 always proposes $(1 - x_2, x_2)$ and accepts a proposal y_1 iff $y_1 \geq \delta_1(1 - x_2)$,
  - player 2 always proposes $(\delta_1(1 - x_2), 1 - \delta_1(1 - x_2))$ and accepts a proposal y_2 iff $y_2 \geq x_2$.

The Baron-Ferejohn legislative bargaining model

- There are three legislators, one of whom is randomly recognized (chosen) to introduce a motion about how to divide a pie of size 1 (think of the pie as cabinet positions).
- Under the *closed rule*, the legislature must vote on the motion without amendments.
- Under the *open rule*, after a legislator introduces a motion, one of the remaining legislators is randomly recognized, and she can either *move the motion* to the vote, or *offer an amendment* (her own motion).
- A motion passes if a majority vote for it. Otherwise the whole process starts again. Infinite horizon game. Common discount factor δ.
- We consider **stationary strategies**, which require a player to take the same action in structurally equivalent subgames.

Closed rule: setup

- First, consider the closed rule. C stands for chance, and V stands for a vote.



- Suppose legislator 1 is recognized, how would she propose to divide the pie?

Closed rule: solution

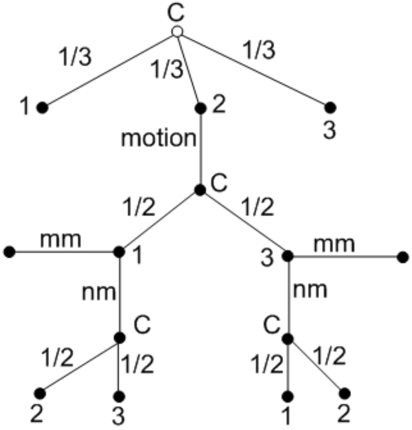
- Let player 1's proposal be (x_1, x_2, x_3) s.t. $x_1 + x_2 + x_3 = 1$.
- Majority voting: player 1 just needs one more supporter for the motion. Suppose she targets player 2 for support, and then her proposal would be in the form $(1 - x, x, 0)$.
- For this to be equilibrium proposal (so player 2 will accept this proposal), x has to be equal or greater than 2's expected payoff if she rejects the proposal:

$$x \geq \delta(\frac{1-x}{3} + \frac{x}{3}) \Rightarrow x \geq \delta/3.$$

  - All players are identical, so in equilibrium player 2 uses the same strategy.
- So the optimal strategy of whoever is the proposer is to offer $\delta/3$ to one of the other players, and other players vote for the motion if and only if they receive an offer of at least $\delta/3$.
  - You can use the one-deviation property to verify that this is indeed a SPNE.

Open rule: setup

- Next we consider the open rule. Here "mm" stands for "move the motion" and "nm" stands for "new motion".



- A player will "mm" only if she supports the motion and the motion will pass the vote.

Open rule: joint inclusion strategy

- Say player 2 is recognized. She can propose $(x, 1-2x, x)$ to have whoever is recognized next to mm (**joint inclusion**), or to make an offer to (say) player 1 alone, i.e., $(y, 1-y, 0)$ (**selective inclusion**).
- If the proposer chooses joint inclusion, i.e., the offer is $(x, 1-2x, x)$, then for the other two players to accept it, it must be that

$$x \geq \delta(1-2x) \Rightarrow x \geq \delta/(1+2\delta).$$

The proposer gets

$$1-2x = 1/(1+2\delta).$$

Open rule: selective inclusion strategy

- If the proposer uses the selective inclusion offer $(y, 1-y, 0)$, let's assume that when a proposer excludes a member in her proposal, that member will not include the earlier proposer if the latter subsequently becomes the proposer.
- Let v_p be the value of being a proposer, and v_e the value of being an excluded member. A proposer has to pay another member δv_p to get her support.
- What is v_p for player 2 when she adopts the selective inclusion strategy and only seek player 1's support? There is a .5 chance that 1 will be recognized following 2's proposal, in which case 2's proposal gets a "mm" and she gets $(1-\delta v_p)$; with the other .5 probability player 3 is recognized, in which case player 2 gets v_e. So

$$v_p = \frac{1}{2}(1-\delta v_p) + \frac{1}{2}\delta v_e.$$

Open rule: selective inclusion strategy (cont.)

- What is v_e? An excluded member has a .5 chance of being recognized in one period's time. So

$$v_e = \frac{1}{2}\delta v_p.$$

- Combining the two equations we get $v_p = 2/(4 + 2\delta - \delta^2)$.
- So the equilibrium selective inclusion offer is $1 - v_p = (2 + 2\delta - \delta^2)/(4 + 2\delta - \delta^2)$.
- Simple algebra shows that selective inclusion is better for a proposer if $\delta > \delta^* \equiv \sqrt{3} - 1$.
- Features of the open rule model:
  - When $\delta < \delta^*$ the coalition is greater than minimal winning.
  - When $\delta > \delta^*$ there can be equilibrium delay in agreement.